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LINEARIZED STABILITY IMPLIES ASYMPTOTIC STABILITY FOR RADIALLY SYMMETRIC EQUILIBRIA OF p-LAPLACIAN BOUNDARY VALUE PROBLEMS IN THE UNIT BALL IN $\mathbb{R}^N$

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ABSTRACT. We consider the parabolic initial-boundary value problem

$$\begin{gathered} \frac{\partial v}{\partial t} = \Delta_p(v) + f(|x|, v), \quad \text{in } \Omega \times (0, \infty), \\ v = 0, \quad \text{in } \partial\Omega \times [0, \infty), \\ v = v_0 \in C_0^0(\overline{\Omega}), \quad \text{in } \overline{\Omega} \times \{0\}, \end{gathered}$$

where $\Omega = B_1$ is the unit ball centered at the origin in $\mathbb{R}^N$, with $N \geq 2$, $p > 1$, and Δ_p denotes the p-Laplacian on Ω. The function $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is continuous, and the partial derivative f_v exists and is continuous and bounded on $[0, 1] \times \mathbb{R}$. It will be shown that (under certain additional hypotheses) the 'principle of linearized stability' holds for radially symmetric equilibrium solutions u_0 of the equation. That is, the asymptotic stability, or instability, of u_0 is determined by the sign of the principal eigenvalue of a linearization of the problem at u_0. It is well-known that this principle holds for the semilinear case $p = 2$ (Δ_2 is the linear Laplacian), but has not been shown to hold when $p \neq 2$. We also consider a bifurcation type problem similar to the one above, having a line of trivial solutions and a curve of non-trivial solutions bifurcating from the line of trivial solutions at the principal eigenvalue of the p-Laplacian. We characterize the stability, or instability, of both the trivial solutions and the non-trivial bifurcating solutions, in a neighbourhood of the bifurcation point, and we obtain a result on 'exchange of stability' at the bifurcation point, analogous to the well-known result when $p = 2$.

1. Introduction

We consider the parabolic initial-boundary value problem

$$\begin{gathered} \frac{\partial v}{\partial t} = \Delta_p(v) + f(|x|, v), \quad \text{in } \Omega \times (0, \infty), \\ v = 0, \quad \text{in } \partial\Omega \times [0, \infty), \\ v = v_0 \in C_0^0(\overline{\Omega}), \quad \text{in } \overline{\Omega} \times \{0\}, \end{gathered} \tag{1.1}$$

where: $\Omega = B_1$ is the unit ball centered at the origin in $\mathbb{R}^N$, with $N \geq 2$; $p > 1$ and Δ_p denotes the p-Laplacian, that is, $\Delta_p(v) = \operatorname{div}(|\nabla v|^{p-2}\nabla v)$ on Ω, where $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^N$; $C_0^0(\overline{\Omega})$ denotes the set of continuous functions

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on Ω satisfying the Dirichlet boundary conditions. The function f is assumed to satisfy the following hypothesis:

$$f : [0,1] \times \mathbb{R} \to \mathbb{R} \text{ is continuous and the partial derivative } f_\xi(r,\xi) \text{ exists on } [0,1] \times \mathbb{R} \text{ and is continuous and bounded.} \tag{1.2}$$

We are only interested in bounded solutions of (1.1), so the boundedness assumption in (1.2) is not restrictive. Under these hypotheses it is known that for any $v_0 \in C_0^0(\overline{\Omega})$ problem (1.1) has a unique solution $t \to v_{v_0}(t) : [0,\infty) \to C_0^0(\overline{\Omega})$, see Theorem 5.2 below (the definition of a 'solution' of (1.1) will be made precise in Definition 5.1).

We wish to determine the dynamic, asymptotic stability, or instability, of equilibrium solutions of (1.1) in terms of their 'linearized stability'. Specifically, suppose that u_0 is an equilibrium solution of (1.1). Then u_0 is asymptotically stable if, for all v_0 sufficiently close to u_0 (in a suitable sense) the corresponding solution $v_{v_0}(t) \to u_0$, in $C_0^0(\overline{\Omega})$, as $t \to \infty$; u_0 is unstable if there exists initial conditions v_0 arbitrarily close to u_0 for which the solution $v_{v_0}(t)$ moves away from u_0 (this will be made more precise in Theorem 6.1 below). In the semilinear case $p = 2$ (where Δ_2 is the standard, linear Laplacian), in a general, bounded domain $\Omega \subset \mathbb{R}^N$, with $N \geq 1$ and smooth boundary $\partial\Omega$, it is well-known that the asymptotic stability, or instability, of u_0 is determined by the location of the spectrum of the linearization of the operator on the right hand side of (1.1) at u_0; this is the so-called 'principle of linearized stability'. See, for example, [12]. However, this principle has not been shown to hold for the general, quasilinear, p-Laplacian problem (1.1) when $1 < p \neq 2$.

In this article we consider the general, quasilinear case $1 < p \neq 2$, but we only consider radially symmetric equilibria on the ball B_1 in $\mathbb{R}^N$, $N \geq 2$. We first define a 'linearization' operator for the right-hand side of (1.1) at a suitable radially symmetric equilibrium u_0. The difficulty here lies in linearising the p-Laplacian operator, and this will require a considerable amount of preparation. The linearization operator will turn out to be a Sturm-Liouville differential operator, and we will say that u_0 is 'linearly stable' (respectively, 'linearly unstable') if the principal eigenvalue $\sigma_0(u_0)$ of the linearization at u_0 is negative (respectively, positive). Of course, until we know that a 'principle of linearized stability' holds for this problem, linear stability or instability tells us nothing about the dynamics of the problem. However, we then show that, under certain additional conditions, linear stability (respectively, linear instability) implies asymptotic stability (respectively, instability) of u_0. In addition, it will be shown that the rate of convergence to u_0 (or divergence from u_0) is at least exponential, with rate determined by $|\sigma_0(u_0)|$. Thus we obtain a close analogue of the semilinear results in [12].

It should be emphasized that although we only consider radially symmetric equilibria u_0, we do not assume that the initial condition v_0 in (1.1) is radially symmetric, so the time-dependent solutions of (1.1) that we consider need not be radially symmetric. That is, we are genuinely considering stability of u_0 with respect to all nearby solutions (in a suitable sense).

The problem (1.1) was considered in [17] in the 1-dimensional case (where $N = 1$ and $\Omega \subset \mathbb{R}$ was an interval), and the results obtained here for the ball $B_1 \subset \mathbb{R}^N$ are similar to the results obtained in [17] (and indeed, they are similar to the results obtained in [12] for the semilinear case). In fact, we obtain the desired linearization operator here by using radial symmetry to reduce the problem on the ball B_1 to an ODE problem on the interval $[0, 1)$, and then using differentiability properties of the inverse of the 1-dimensional, radially symmetric form of the p-Laplacian operator obtained from this reduction to construct the linearization operator. The required differentiability properties of the 1-dimensional, radially symmetric form of the p-Laplacian were obtained in [4, 11, 15]. Unfortunately, these differentiability results have not been extended to general, non-radially symmetric domains in higher dimensions, so the results here cannot readily be extended to such domains. A stability result for general, smooth domains in $\mathbb{R}^N$ is obtained in [17, Theorem 5.7], using some of the machinery in [17], but it is not a linearized stability result (it is weaker than that), and does not use any differentiability properties of the p-Laplacian.

A detailed discussion and comparison of these linearized stability results with related results from the literature was given in [17, Section 5.1]. Although the results obtained in [17] only dealt with the case $N = 1$, the discussion in [17] also described related results from the literature for the case $N > 1$, so in fact this discussion applies equally well to the results obtained here in the ball $B_1 \subset \mathbb{R}^N$. Hence, we will not repeat this discussion here and merely refer to [17] for this.

Finally, in Section 7, we consider a bifurcation type problem similar to (1.1), having a line of trivial solutions, and a curve of non-trivial solutions bifurcating from the line of trivial solutions at the principal eigenvalue of the p-Laplacian. We characterise the stability, or instability, of both the trivial solutions and the non-trivial bifurcating solutions, in a neighbourhood of the bifurcation point, and we obtain a result on 'exchange of stability' of these solutions at the bifurcation point, analogous to the well-known result when $p = 2$, see [9, 12].

2. Preliminaries

2.1. **Notation.** For $q \geq 1$, $L^q(\Omega)$ will denote the standard space of real valued functions on Ω whose qth power is integrable, with norm $\|\cdot\|_q$ (throughout, all function spaces will be real); the standard $L^2(\Omega)$ inner product will be denoted by $\langle\cdot,\cdot\rangle$; $W^{1,q}(\Omega)$ will denote the standard Sobolev space of functions on $\overline{\Omega}$ whose first order derivative belongs to $L^q(\Omega)$, with norm $\|\cdot\|_{1,q}$, and its dual space will be denoted by $W^{-1,p'}(\Omega)$, where $p' := p/(p-1)$ is the conjugate exponent of p. For $j = 0, 1, \dots$, $C^j(\overline{\Omega})$ will denote the standard space of j times continuously differentiable functions defined on $\overline{\Omega}$, with the standard sup-norm $|\cdot|_j$. For any $\omega_0 \in C^j(\overline{\Omega})$,

$$B^j_r(\omega_0) := \{\omega \in C^j(\overline{\Omega}) : |\omega - \omega_0|_j < r\}, \quad r > 0.$$

We also let $C^j_0(\overline{\Omega})$ and $W^{1,q}_0(\Omega)$ denote the sets of functions ω in $C^j(\overline{\Omega})$ and $W^{1,q}(\Omega)$, respectively, satisfying the boundary conditions $\omega = 0$ on $\partial\Omega$, and $B^j_{0,r}(\omega_0) := B^j_r(\omega_0) \cap C^j_0(\overline{\Omega})$.

If $h : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is continuous then, for any $\omega \in C^0(\overline{\Omega})$, we define $h(\omega) \in C^0(\overline{\Omega})$ by

$$h(\omega)(x) := h(x, \omega(x)), \quad x \in \overline{\Omega}.$$

Clearly, the 'Nemytskii' operator $\omega \to h(\omega) : C^0(\overline{\Omega}) \to C^0(\overline{\Omega})$ is continuous. In particular, we define the function $\phi_p(\xi) := |\xi|^{p-1} \operatorname{sgn} \xi$, $\xi \in \mathbb{R}$, with the corresponding Nemytskii operator $\phi_p : C^0(\overline{\Omega}) \to C^0(\overline{\Omega})$. We also note that it follows from the assumption (1.2) that the Nemytskii operator associated with the function f in

(1.1) is in fact C^1, with Fréchet derivative at arbitrary $u_0 \in C^0(\overline{\Omega})$ given by

$$Df(u_0)\bar{u} = f_\xi(u_0)\bar{u}, \quad \bar{u} \in C^0(\overline{\Omega}). \tag{2.1}$$

2.2. **p-Laplacian.** We define $\Delta_p : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ as follows: for any $\omega \in W_0^{1,p}(\Omega)$, we define $\Delta_p(\omega) \in W^{-1,p'}(\Omega)$ by

$$\int_\Omega \Delta_p(\omega)\, \psi := -\int_\Omega |\nabla\omega|^{p-2}\nabla\omega \cdot \nabla\psi, \quad \forall \psi \in W_0^{1,p}(\Omega). \tag{2.2}$$

A precise definition of what is meant by a solution of (1.1) will be given in Section 5.1 below.

3. Radial p-Laplacian

We will use the notation $r := |x|$, $x \in \mathbb{R}^N$, and a function of the form $x \to u(r) = u(|x|)$, $x \in \Omega$, where $u : [0,1) \to \mathbb{R}$, will be called a *radial* function on Ω. For simplicity, we will use the same notation for a radial function, defined on Ω, and the corresponding function defined on $[0,1)$; this should not cause any confusion. For a smooth radial function u on Ω, the p-Laplacian $\Delta_p(u)$ has the form

$$\Delta_p(u) = \operatorname{div}(|\nabla u|^{p-2}\nabla u) = \frac{1}{r^{N-1}}\big(r^{N-1}|u'|^{p-2}u'\big)' = \frac{1}{r^{N-1}}\big(r^{N-1}\phi_p(u')\big)', \tag{3.1}$$

on Ω (at least, at values of $r > 0$ where $u'(r) \neq 0$), where, in this context, r^{N-1} denotes the function $x \to r^{N-1}$ on $\overline{\Omega}$ and u' denotes the standard, 1-dimensional derivative of the function $u : [0,1) \to \mathbb{R}$.

For a given function $h \in C^0[0,1]$ we now consider the boundary value problem

$$\begin{gathered} \Delta_p(u) = h, \quad \text{on } \Omega, \\ u = 0, \quad \text{on } \partial\Omega \end{gathered} \tag{3.2}$$

(regarding h as a radial function on Ω, and with $\Delta_p(u)$ interpreted in the (weak) sense of (2.2)), and we search for a radial solution u. By (3.1), this reduces to solving the one-dimensional boundary value problem

$$\begin{gathered} \big(r^{N-1}\phi_p(u')\big)' = r^{N-1}h, \quad \text{on } (0,1), \\ u'(0) = u(1) = 0. \end{gathered} \tag{3.3}$$

The following result is proved in [11, Section 3].

Theorem 3.1. *For $1 < p \neq 2$, and any $h \in C^0[0,1]$, problem* (3.3) *has a unique solution $u = S_p(h) \in C^1[0,1]$. The operator $S_p : C^0[0,1] \to C^1[0,1]$ is continuous.*

We will denote by $\mathcal{D}_p$ the range of the operator $S_p : C^0[0,1] \to C^1[0,1]$ (in a sense, $\mathcal{D}_p$ is a 'domain' for an operator formulation of problem (3.3)). Clearly,

$$\mathcal{D}_p \subsetneq \{u \in C^1[0,1] : \ u'(0) = u(1) = 0 \text{ and } r^{N-1}\phi_p(u') \in C^1[0,1]\}.$$

The next result follows readily from Theorem 3.1, although it was not proved in [11] (it was not needed in [11], but is needed here).

Corollary 3.2. *The solution $u = S_p(h) \in \mathcal{D}_p$ of* (3.3) *found in Theorem 3.1 is a radial solution of* (3.2). *A solutions of* (3.2) *is unique (by the monotonicity of the p-Laplacian operator Δ_p) so u is the unique solution of* (3.2).

Remark 3.3. (i) The operator S_p as defined here has the opposite sign to the operator S_p defined in [11]. The sign that we use here ensures that the spectrum of the linearization of S_p, which we define below, is better suited to determining the dynamics of the time-dependent problem (1.1) than the sign used in [11], in the sense that, with our sign convention, a negative principal eigenvalue corresponds to stability, which is the usual sign convention for linearized stability.

(ii) An explicit formula for the operator S_p is given in [11, Section 3] in terms of integral operators and the Nemytskii operator $\phi_{p^*+1} = \phi_p^{-1}$, where $p^* := (p-1)^{-1}$. This explicit formula is relatively easy to construct but we will omit it here. The explicit form for S_p shows that the solution $u = S_p(h)$ of (3.3) is C^1 at $r = 0$, and hence $u \in C^1[0,1]$, which is perhaps not immediately obvious from (3.3).

3.1. **Differentiability of S_p.** We now state some differentiable properties of the operator S_p.

Theorem 3.4 ([11, Theorem 3.5])**.** *For any $h \in C^0[0,1]$, let $u(h) = S_p(h) \in C^1[0,1]$.*

(A) Suppose that $1 < p < 2$. Then $S_p : C^0[0,1] \to C^1[0,1]$ is C^1, and for all $h, \bar{h} \in C^0[0,1]$, the derivative $w = DS_p(h)\bar{h} \in C^1[0,1]$ is given by

$$w(r) = -p^* \int_r^1 \Big\{ |u(h)'(s)|^{2-p} \int_0^s \big(\frac{t}{s}\big)^{N-1} \bar{h}(t)\, dt \Big\}\, ds, \quad r \in [0,1], \tag{3.4}$$

and w has the following properties

$$\begin{gathered} r^{N-1}|u(h)'|^{p-2}w' \in C^1[0,1], \\ (p-1)(r^{N-1}|u(h)'|^{p-2}w')' = r^{N-1}\bar{h}, \\ w'(0) = w(1) = 0. \end{gathered} \tag{3.5}$$

(B) Suppose that $p > 2$ and $h_0 \in C^0[0,1]$ is such that

$$u(h_0)'(x) = 0 \implies h_0(x) \neq 0, \quad x \in [0,1]. \tag{3.6}$$

Then there exists a neighbourhood V_0 of h_0 in $C^0[0,1]$ such that:

(a) *$h \in V_0 \implies |u(h)'|^{2-p} \in L^1(0,1)$, and the mapping $h \to |u(h)'|^{2-p} : V_0 \to L^1(0,1)$ is continuous;*
(b) *the operator $S_p : V_0 \subset C^0[0,1] \to W^{1,1}(0,1)$ is C^1, and is again given by* (3.4)*;*
(c) *the properties of w in* (3.5) *hold for any $h \in V_0$, $\bar{h} \in C^0[0,1]$, with $w \in W^{1,1}(0,1)$ (instead of $w \in C^1[0,1]$). To clarify this* (*in particular, the boundary condition $w'(0) = 0$*):
   - *the zeros of $u(h)'$ on $[0,1]$ are isolated (by* (3.3) *and* (3.6)*);*
   - *w' exists and is continuous on $(0,1]$, except possibly at the zeros of $u(h)'$ (by* (3.5)*);*
   - *$w' \in C^0[0,\epsilon]$, for some $\epsilon \in (0,1]$, and $w'(0) = 0$.*

Proof. The above result is proved in [11] (bearing in mind the opposite sign used here compared to [11], as mentioned in Remark 3.3 above). The results in the 'clarification' in part (c) are not explicitly stated in [11], but they are, essentially, obtained in the proof there – this is described, with some additional details, in the proof of [15, Theorem 2.4]. □

Remark 3.5. The range space in part (A) of Theorem 3.4 is $C^1[0,1]$, while in part (B) it is $W^{1,1}(0,1)$. This slight loss of regularity in part (B) was the reason for the 'clarification' regarding the sense in which the boundary condition $w'(0)=0$ in (3.5) holds (a priori, for $w \in W^{1,1}(0,1)$ the value of $w'(0)$ does not seem to be well-defined). From now on, when $p>2$ we will regard the boundary condition $w'(0)=0$ in (3.5) as holding in the sense described in Theorem 3.4 (B).

The loss of regularity in part (B) of Theorem 3.4 will also cause a further difficulty below, and a slight extension of this result will be required to deal with this. For any $\epsilon \in (0,1]$, let

$$E_\epsilon := [1-\epsilon, 1].$$

For $j=0,1$, the space $C^j(E_\epsilon)$ has the obvious meaning, and its norm will be denoted by $|\cdot|_{j,\epsilon}$. We now define a 'restriction' operator $P_\epsilon : C^1[0,1] \to C^1(E_\epsilon)$ by

$$P_\epsilon g := g|_{E_\epsilon}, \quad g \in C^1[0,1].$$

Theorem 3.6 ([17, Theorem 3.6])**.** *Suppose that $p>2$ and $h_0 \in C^0[0,1]$ is such that $u(h_0)'(1) \neq 0$. Then there exists $\epsilon>0$ and a neighbourhood V_0 of h_0 in $C^0[0,1]$ such that the mapping $P_\epsilon \circ S_p : V_0 \to C^1(E_\epsilon)$ is C^1.*

Proof. The proof follows the proof in [17]. Although the proof in [17] is for the case $N=1$, the term r^{N-1} here essentially changes nothing near to $r=1$ so the same proof works here. □

3.2. **Inverse of $DS_p(h_0)$.** For a given $h \in C^0[0,1]$ the operator $DS_p(h)$ in (3.4) is an integral operator. It will be convenient to discuss the corresponding differential operator on $(0,1)$, which will enable us to ascertain, and utilize, some of the spectral properties of $DS_p(h)$.

We consider a fixed $h_0 \in C^0[0,1]$, and $u_0 := S_p(h_0) \in \mathcal{D}_p$, such that the following conditions hold:

$$\begin{gathered}\text{if } 1<p<2 \text{ then } u_0' \neq 0 \text{ a.e. on } (0,1);\\ \text{if } p>2 \text{ then the hypothesis (3.6) in Theorem 3.4 } (B) \text{ holds.}\end{gathered} \tag{3.7}$$

If $p>2$ then, by Theorem 3.4 (B), $u_0' \neq 0$ a.e. on $(0,1)$, so if (3.7) holds then $u_0' \neq 0$ a.e. on $(0,1)$ for all $1<p\neq 2$.

From the above results, we now summarize some properties of the elements of the range $R(DS_p(h_0))$, which hold for all $1<p\neq 2$. For arbitrary $\bar{h} \in C^0[0,1]$, let $w = DS_p(h_0)\bar{h} \in R(DS_p(h_0))$. Then, by Theorem 3.4 and (3.7),

$$\begin{gathered} w \in W^{1,1}(0,1), \quad |u_0'|^{p-2}w' \in C^1[0,1],\\ (p-1)r^{1-N}\big(r^{N-1}|u_0'|^{p-2}w'\big)' = \bar{h}, \text{on } (0,1],\\ w' \in C^0[0,\epsilon], \quad \text{for some } \epsilon \in (0,1], \text{ and } w'(0)=w(1)=0. \end{gathered} \tag{3.8}$$

In view of these properties we may make the following definition.

Definition 3.7. For any $h_0 \in C^0[0,1]$ and $u_0 = S_p(h_0) \in \mathcal{D}_p$ satisfying (3.7) we define the linear operator $\Lambda_{p,u_0} : \mathcal{D}(\Lambda_{p,u_0}) \to C^0[0,1]$ as follows:

$$\mathcal{D}(\Lambda_{p,u_0}) := R(DS_p(h_0)), \tag{3.9}$$

$$\Lambda_{p,u_0} w := (p-1)r^{1-N}\big(r^{N-1}|u_0'|^{p-2}w'\big)' \quad \text{on } (0,1],\ w \in \mathcal{D}(\Lambda_{p,u_0}). \tag{3.10}$$

Remark 3.8. By definition, Λ_{p,u_0} is a differential operator of Sturm-Liouville type. The coefficient $|u_0'|^{p-2}$ in this operator satisfies $1/|u_0'|^{p-2} = |u_0'|^{2-p} \in L^1(0,1)$ (by Theorem 3.4 (B) (a), when $p > 2$), which is a standard hypothesis in the theory of Sturm-Liouville differential operators, see [2, Chap. 8] or [6, Chaps. 1-2]. Unfortunately, the coefficients r^{1-N}, r^{N-1} in the operator Λ_{p,u_0} do not fall so readily within the standard theory. Our main goal here is to construct sub and super-solutions of the differential equation (1.1), for which we will require both the differential operator formulation in (3.10), and also the bounded, integral operator $DS_p(h_0)$, to enable us to use the implicit function theorem to construct the required sub and super-solutions. For this, we will require some information about the eigenvalues of Λ_{p,u_0} which, for the above reasons, does not follow immediately from the standard Sturm-Liouville theory (as in [2, 6]) so we will derive it directly below.

It follows from Definition 3.7, together with the properties of $DS_p(h_0)$ in (3.8), that

$$\Lambda_{p,u_0} DS_p(h_0)\bar{h} = \bar{h}, \quad \bar{h} \in C^0[0,1], \quad DS_p(h_0)\Lambda_{p,u_0}\bar{w} = \bar{w}, \quad \bar{w} \in \mathcal{D}(\Lambda_{p,u_0}), \tag{3.11}$$

$$u_0 \in \mathcal{D}(\Lambda_{p,u_0}), \quad \Lambda_{p,u_0} u_0 = (p-1)\Delta_p(u_0), \tag{3.12}$$

$$\langle r^{N-1}\Lambda_{p,u_0} w_1, w_2\rangle = \langle r^{N-1} w_1, \Lambda_{p,u_0} w_2\rangle, \quad w_1, w_2 \in \mathcal{D}(\Lambda_{p,u_0}). \tag{3.13}$$

Here, the right-hand side of the equation in (3.12) is given by (3.1), while (3.13) follows from the definition of Λ_{p,u_0} and two integrations by parts (the properties of the functions $w \in \mathcal{D}(\Lambda_{p,u_0})$ described in (3.8) justify these integrations).

4. A linearization operator at radial equilibria of (1.1)

Naturally, an *equilibrium* of (1.1) is a solution u of the problem

$$\begin{gathered} \Delta_p(u) + f(u) = 0, \quad \text{on } \Omega, \\ u = 0, \quad \text{on } \partial\Omega, \end{gathered} \tag{4.1}$$

which we regard as a constant (in time) solution of (1.1). Recalling (3.1), if an equilibrium u is radial then it satisfies the 1-dimensional problem

$$\begin{gathered} \big(r^{N-1}\phi_p(u')\big)' + r^{N-1} f(u) = 0, \quad \text{on } (0,1), \\ u'(0) = u(1) = 0, \end{gathered} \tag{4.2}$$

and hence, by Theorem 3.1, $u \in \mathcal{D}_p$ and $u + S_p(f(u)) = 0$.

Remark 4.1. Potentially, not all solutions of (4.1) are radial, but these are the ones we will be considering. Under the additional assumption that $f(r,\xi)$ is decreasing in $r \in [0,1]$, it follows from [5, Theorem 1, p. 51] that any positive solution $u \in W_0^{1,p}(\Omega) \cap C^1(\overline{\Omega})$ of (4.1) is radial.

We now wish to define a 'linearization' of (4.1) at radial equilibria. Suppose that u_0 is a solution of (4.2) which satisfies the following conditions (recall that $\mathcal{D}_p \subset C^1[0,1]$):

$$u_0'(1) \neq 0, \quad u_0' \neq 0 \text{ a.e. on } (0,1); \tag{4.3}$$

$$\text{if } p > 2 \text{ then } u_0'(x) = 0 \implies f(u_0(x)) \neq 0, \quad x \in [0,1]. \tag{4.4}$$

These conditions ensure that condition (3.7) holds (with $h_0 = -f(u_0)$), so that the operator Λ_{p,u_0} in Definition 3.7 is well-defined, and hence we may make the following definition.

Definition 4.2. Suppose that u_0 satisfies (4.2)–(4.4). Then the *linearization* of (4.2) at u_0 is defined to be the operator

$$\mathcal{L}_{u_0} w := \Lambda_{p,u_0} w + f_\xi(u_0) w, \quad w \in \mathcal{D}(\Lambda_{p,u_0}).$$

We will require some basic properties of the eigenvalues of $\mathcal{L}_{u_0}$ which, as mentioned in Remark 3.8, are difficult to obtain from standard Sturm-Liouville theory, so we will derive them directly. We first define a principal eigenvalue and eigenfunction of $\mathcal{L}_{u_0}$, which will be used in our stability results.

Definition 4.3. Suppose that u_0 satisfies (4.2)-(4.4). Then $\sigma \in \mathbb{R}$ is an eigenvalue of $\mathcal{L}_{u_0}$, with corresponding eigenfunction ψ, if $0 \neq \psi \in \mathcal{D}(\Lambda_{p,u_0})$ and $\mathcal{L}_{u_0}\psi = \sigma\psi$. In addition, σ is a *principal eigenvalue* if it has an eigenfunction $\psi \geq 0$ in $[0,1]$.

The next result follows immediately from the above definitions and (3.11). We note that the operator S_p is odd, so that $DS(-f(u_0)) = -DS(f(u_0))$, and we will use the notation $DS_p(f(u_0)) \circ f_\xi(u_0)$ for the operator $w \to DS_p(f(u_0))(f_\xi(u_0)w)$.

Lemma 4.4. *A number σ is an eigenvalue of $\mathcal{L}_{u_0}$, with eigenfunction ψ, if and only if*

$$\big(I + DS_p(f(u_0)) \circ f_\xi(u_0)\big)\psi = \sigma DS_p(f(u_0))\psi, \quad 0 \neq \psi \in C^0[0,1]. \tag{4.5}$$

We now prove the existence and uniqueness of a principal eigenvalue of $\mathcal{L}_{u_0}$.

Theorem 4.5. *The operator $\mathcal{L}_{u_0}$ has a unique principal eigenvalue σ_0.*

Proof. To prove the existence of a principal eigenvalue σ_0 we first consider the integral equation

$$\tilde{w}(r) = 1 + p^* \int_0^r \Big\{ |u_0'(s)|^{2-p} \int_0^s \big(\frac{t}{s}\big)^{N-1} (\lambda - f_\xi(u_0(t)))\tilde{w}(t)\,dt \Big\}\,ds \tag{4.6}$$

for $r \in [0,1]$, where $\lambda \in \mathbb{R}$. To motivate considering this equation, and for use below, we first show that if $\tilde{w} \in C^0[0,1]$ satisfies (4.6) then it also satisfies the initial value problem

$$\begin{gathered}(p-1)r^{1-N}\big(r^{N-1}|u_0'|^{p-2}\tilde{w}'\big)' = (\lambda - f_\xi(u_0))\tilde{w}, \quad \text{on } (0,1],\\ \tilde{w}(0) = 1, \quad \tilde{w}'(0) = 0,\end{gathered} \tag{4.7}$$

in the sense that $\tilde{w}$ has similar properties to those described in (3.8), with $\bar{h} = (\lambda - f_\xi(u_0))\tilde{w}$ and with the boundary condition $\tilde{w}(1) = 0$ replaced by the initial condition $\tilde{w}(0) = 1$. This claim would be trivial if it were not for the potentially singular behaviour near to $r = 0$, but in view of this we will briefly sketch the argument.

We first note that by [11, Lemma 3.2] the inner integral operator in (4.6) maps $C^0[0,1]$ into $C^1[0,1]$, and since $|u_0'|^{2-p} \in L^1(0,1)$ (see Remark 3.8), the overall integral operator on the right hand side of (4.6) defines a bounded linear operator from $C^0[0,1]$ into $W^{1,1}(0,1)$. In fact, the integral operator in (4.6) clearly has a similar form to that of the integral operator $DS_p(h_0)$, as described in (3.4), although there is a slight difference due to the different boundary conditions for the operator S_p (see (3.5)) and in problem (4.7). In view of this it can be seen that any solution of the integral equation (4.6) satisfies the differential equation in (4.7), on $(0,1]$, and the boundary condition $\tilde{w}(0)=1$.

To see that the boundary condition $\tilde{w}'(0)=0$ also holds we first note that, by Theorem 3.4, the coefficient function $|u_0'|^{2-p}$ is continuous on an interval $(0,\epsilon]$, for some $\epsilon\in(0,1]$. Hence, we can differentiate (4.6) to obtain a formula for $\tilde{w}'$ on $(0,\epsilon]$. This formula shows that $\tilde{w}'$ is continuous on $(0,\epsilon]$, and is the same as the formula for v' in (3.10) in [11] (apart from the sign). Now, the argument following (3.10) in [11] (together with the slight extension of this argument at the end of the proof of [15, Theorem 2.4]) shows that $\lim_{r\to 0^+}\tilde{w}'(r)=0$, and so, by l'Hôpital's rule, $\tilde{w}'(0)$ exists and $\tilde{w}'(0)=0$, so that $\tilde{w}'\in C^0[0,\epsilon]$, which completes the proof that $\tilde{w}'(0)=0$.

Now, the standard proof of the Picard existence theorem for ODE initial value problems (see Chapters 1-2 of [6] or [18, Proposition 1.8]) can readily be adapted to show that there exists $\beta\in(0,1]$ such that (4.6) has a unique solution $\tilde{w}\in W^{1,1}(0,\beta)$ on the interval $[0,\beta)$; once away from $r=0$ this solution can then be extended to the interval $[0,1]$ by standard ODE theory (see [6]), which applies to the ODE in (4.7) everywhere except near to $r=0$. We will denote this solution by $\tilde{w}_\lambda$. It can also be shown that the mapping $\lambda\to\tilde{w}_\lambda:\mathbb{R}\to C^0[0,1]$ is continuous (by the methods of Chapters 1-2 of [6], or by an adaptation of the proof of part (c) of [18, Proposition 1.8]).

We now show that there exists σ_0 such that $\tilde{w}_{\sigma_0}(1)=0$. It is clear from (4.6) that there exists $L>0$ such that $\lambda\ge L\implies\tilde{w}_\lambda>0$ on $[0,1]$, and $\lambda\le -L\implies\tilde{w}_\lambda$ has a zero on $(0,1)$. Now let

$$\sigma_0:=\inf\{\lambda:\tilde{w}_\mu>0 \text{ on } [0,1] \text{ for all } \mu>\lambda\}.$$

By continuity, $\tilde{w}_{\sigma_0}\ge 0$ on $[0,1]$ and $\tilde{w}_{\sigma_0}(r_0)=0$, for some $r_0\in(0,1]$. These properties imply that if $r_0\in(0,1)$ then r_0 is a double zero of $\tilde{w}_{\sigma_0}$ and hence, by standard ODE theory, $\tilde{w}_{\sigma_0}\equiv 0$. However, this contradicts the initial condition $\tilde{w}_{\sigma_0}(0)=1$, so we must have $\tilde{w}_{\sigma_0}(1)=0$, which is what we wanted. We will now write $\psi_0=\tilde{w}_{\sigma_0}$.

Substituting $\lambda=\sigma_0$ and $\tilde{w}=\psi_0$ into (4.6), and setting $r=1$, shows that

$$1+p^*\int_0^1\{\dots\}\,ds=0,$$

where the argument $\{\dots\}$ inside the integration is as in (4.6) (with these substitutions), and hence, again by (4.6),

$$\psi_0(r)=-p^*\int_0^1\{\dots\}\,ds+p^*\int_0^r\{\dots\}\,ds=-p^*\int_r^1\{\dots\}\,ds,\quad r\in[0,1].$$

Combining this with Lemma 4.4 shows that σ_0 is an eigenvalue, with eigenfunction ψ_0, which completes the proof of the existence of a principal eigenvalue σ_0.

To prove the uniqueness of the principal eigenvalue σ_0, suppose that $\widetilde{\sigma}_0\ne\sigma_0$ is also a principal eigenvalue, with corresponding principal eigenfunction $\widetilde{\psi}_0\ge 0$. Then, by (3.13),

$$\begin{gathered}\sigma_0\langle r^{N-1}\psi_0,\widetilde{\psi}_0\rangle=\langle r^{N-1}\mathcal{L}_{u_0}\psi_0,\widetilde{\psi}_0\rangle=\langle r^{N-1}\psi_0,\mathcal{L}_{u_0}\widetilde{\psi}_0\rangle=\widetilde{\sigma}_0\langle r^{N-1}\psi_0,\widetilde{\psi}_0\rangle\\ \implies\langle r^{N-1}\psi_0,\widetilde{\psi}_0\rangle=0,\end{gathered}$$

but this is not possible since $\psi_0>0$, and $\widetilde{\psi}_0\ge 0$ and is non-trivial, on $(0,1)$. $\square$

Corollary 4.6. *(a) The principal eigenfunction ψ_0 of $\mathcal{L}_{u_0}$ satisfies $\psi_0 > 0$ on $[0,1)$ and $\psi_0'(1) < 0$.*

(b) All the eigenvalues of $\mathcal{L}_{u_0}$ are real, and if $\sigma \neq \sigma_0$ is an eigenvalue of $\mathcal{L}_{u_0}$ then $\sigma < \sigma_0$.

Proof. (a) The first inequality follows immediately from the construction of ψ_0 in the proof of Theorem 4.5. To prove the second inequality we first note that, by hypothesis (4.3), the coefficient function $|u_0'|^{p-2}$ in the definition of Λ_{p,u_0} in (3.10) is C^1 on an interval $(\epsilon, 1]$, for some $\epsilon \in (0,1)$, so by the form of Λ_{p,u_0} (3.10), any eigenfunction ψ is C^1 near to 1, and so the value of $\psi_0'(1)$ in the inequality is well-defined. The inequality now follows immediately from the other properties in Definition 4.3, together with standard properties of Sturm-Liouville operators (which hold for $\mathcal{L}_{u_0}$ away from $r = 0$).

(b) The reality of the eigenvalues is a standard calculation, using the symmetry property (3.13) of $\mathcal{L}_{u_0}$, similar to the uniqueness proof in the proof of Theorem 4.5 (of course, we should complexify the spaces and operators to make sense of complex eigenvalues, but this argument shows that there is no need to consider this). If $\sigma \neq \sigma_0$ is an eigenvalue then, by uniqueness, σ is not a principal eigenvalue, so the corresponding eigenfunction ψ must have a zero $z \in (0,1)$ and this implies, by standard Sturm-Liouville oscillation theory on the interval $[z,1]$, that $\sigma < \sigma_0$. □

From now on, $\sigma_0 = \sigma_0(u_0)$ will denote the principal eigenvalue of $\mathcal{L}_{u_0}$, and $\psi_0 = \psi_0(u_0)$ will denote the normalised, principal eigenfunction (with $|\psi_0|_0 = 1$). Any function $\omega \in C^0[0,1]$ which is C^1 near to 1 and satisfies $\omega > 0$ on $[0,1)$, $\omega(1) = 0$ and $\omega'(1) < 0$, will be said to be *strongly positive* on $[0,1)$. By Corollary 4.6, the principal eigenfunction ψ_0 is strongly positive. The following result will be useful below.

Lemma 4.7. *If 0 is not an eigenvalue of $\mathcal{L}_{u_0}$ then there exists $\zeta \in \mathcal{D}(\Lambda_{p,u_0})$ such that $\mathcal{L}_{u_0}\zeta \equiv 1$ on $[0,1]$.*

Proof. By (3.11) it suffices to show that the equation

$$\big(I + DS_p(f(u_0)) \circ f_\xi(u_0)\big)\zeta = DS_p(f(u_0))\mathbf{1}, \quad \zeta \in C^0[0,1], \tag{4.8}$$

has a solution ζ, where $\mathbf{1}$ denotes the function $\mathbf{1}(r) = 1$, $r \in [0,1]$. We first note that, since $R(DS_p(f(u_0))) \subset W^{1,1}(0,1)$, the operator

$$K_0 := DS_p(f(u_0)) \circ f_\xi(u_0) : C^0[0,1] \to C^0[0,1]$$

is compact. Also, the null-space $N(I+K_0) = \{0\}$ since, by Lemma 4.4, any element $\psi \neq 0$ of this null-space would be an eigenfunction of $\mathcal{L}_{u_0}$ with eigenvalue $\sigma = 0$, which would contradict the hypothesis that 0 is not an eigenvalue of $\mathcal{L}_{u_0}$. Hence, the operator $I + K_0$ is non-singular, so (4.8) must have a solution ζ. □

5. Solutions of (1.1) and a comparison theorem

We now describe, briefly, the existence, uniqueness and various properties of solutions of the problem (1.1). These results are well-known, and are described in more detail in [17], where references to the preceding literature (from which these results are derived) are also given. In view of this the discussion here is brief.

5.1. **Existence and uniqueness of solutions.** To state precisely what we mean by a solution of problem (1.1) we define the spaces

$$\Sigma_T := C([0,T), C^0_0(\overline{\Omega})) \cap C((0,T), W^{1,p}_0(\Omega)) \cap W^{1,2}_{\rm loc}((0,T), L^2(\Omega)), \quad T>0$$

(in this definition we allow $T=\infty$).

Definition 5.1. A *solution* of (1.1) is a function $v \in \Sigma_T$, for some $T>0$, such that $v(0)=v_0$ and for a.e. $t \in [0,T)$:

(a) the function $v : [0,T) \to L^2(\Omega)$ is differentiable at t;
(b) $\Delta_p(v(t)) \in L^2(\Omega)$ (where $\Delta_p(v(t))$ is defined as in (2.2));
(c) $\frac{dv}{dt}(t) = \Delta_p(v(t)) + f(v(t))$ (in the $L^2(\Omega)$ sense).

Thus, we regard a solution v of (1.1) as a time-dependent mapping $t \to v(t) : [0,T) \to C^0_0(\overline{\Omega})$, satisfying (1.1) in the sense described in Definition 5.1. In view of this we will rewrite (1.1) in the form

$$\frac{dv}{dt} = \Delta_p(v) + f(v), \quad v(0) = v_0 \in C^0_0(\overline{\Omega}). \tag{5.1}$$

Theorem 5.2. *For any $v_0 \in C^0_0(\overline{\Omega})$, the problem* (5.1) *has a unique solution $v_{v_0} \in \Sigma_\infty$.*

The solution v_0 in Theorem 5.2 exists on $[0,\infty)$ due to the boundedness assumption in hypothesis (1.2); without this assumption the solution might 'blow-up' in finite time.

5.2. **Sub and super-solutions of** (5.1) **and a comparison theorem.** We now define sub and super-solutions of (5.1). These need not satisfy the Dirichlet boundary conditions on the boundary $\partial\Omega$, so it will be convenient to define an extension of the operator Δ_p and the solution spaces Σ_T which omits these boundary conditions. Consequently, we define $\tilde\Delta_p : W^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ by supposing that $\omega \in W^{1,p}(\Omega)$ in the definition of Δ_p in (2.2), and we define $\tilde\Sigma_T$ by replacing the spaces $C^0_0(\overline{\Omega})$, $W^{1,p}_0(\Omega)$ with $C^0(\overline{\Omega})$, $W^{1,p}(\Omega)$, respectively, in the definition of Σ_T. We also use the following notation (see [10, p. 1])

$$\Omega_T := \Omega \times (0,T), \quad \Gamma_T := \big(\overline{\Omega} \times \{0\}\big) \cup \big(\partial\Omega \times [0,T)\big), \quad 0 < T \le \infty;$$

the set Γ_T is called the 'parabolic boundary' of Ω_T.

Definition 5.3. Suppose that, for some $T>0$, a function $\tilde v \in \tilde\Sigma_T$ satisfies (a) and (b) in Definition 5.1, with Δ_p replaced by $\tilde\Delta_p$, for a.e. $t \in [0,T)$. Then:

(a) $\tilde v$ is a *sub-solution* of (5.1) on Ω_T if
$$\frac{d\tilde v}{dt}(t) - \tilde\Delta_p(\tilde v(t)) - f(\tilde v(t)) \le 0 \quad \text{(in the } L^2(\Omega) \text{ sense), a.e. } t \in [0,T); \tag{5.2}$$
(b) $\tilde v$ is a *super-solution* of (5.1) on Ω_T if (5.2) holds with the inequality reversed.

Clearly, solutions of (5.1) (including equilibrium solutions) are both sub and super-solutions of (5.1). We now state a comparison theorem for sub and super-solutions of (5.1).

Theorem 5.4. *Suppose that $\tilde u$ is a sub-solution and $\tilde v$ is a super-solution of* (5.1) *on Ω_T, for some $0 < T \le \infty$, and $\tilde u \le \tilde v$ on Γ_T. Then $\tilde u \le \tilde v$ on Ω_T.*

Remark 5.5. Theorem 5.4 is stated and proved in [17, Theorem 4.4] for the case $N = 1$, but both the statement and proof are almost identical for the case $N \geq 2$, so the proof will not be restated here. However, we note that the proof of [17, Theorem 4.4] was based on the proof of [14, Theorem 2.5], which in turn was based on the proof of [10, Lemma 3.1, Ch. VI]; the equation $v_t = \Delta_p(v)$ was considered in [10], while [14] considered the equation $v_t = \Delta_p(v) + \lambda\phi_p(v)$. We also note that Definition 5.3 could be weakened considerably, and the comparison theorem could be generalized, but these suffice for our purposes here.

6. Linearized stability implies asymptotic stability

We suppose throughout this section that u_0 is a radial solution of (4.1) satisfying the conditions of Definition 4.2, and we let $\sigma_0 = \sigma_0(u_0)$, $\psi_0 = \psi_0(u_0)$, denote the principal eigenvalue and eigenfunction of the linearization of (4.1) at u_0, as defined in Definition 4.3. The following result shows that the asymptotic stability, or instability, of u_0 is determined by the sign of σ_0, and that:

- in the stable case, all solutions with initial values $v_0 \in C_0^0(\overline{\Omega})$ sufficiently close to u_0 converge to u_0, exponentially in time with rate determined by the magnitude of σ_0;
- in the unstable case, some solutions diverge away from u_0, exponentially in time (this is made more precise in the theorem).

Theorem 6.1. *Suppose that u_0 is a radial, equilibrium solution of* (5.1) *satisfying the conditions of Definition 4.2, and $v_0 \in C_0^0(\overline{\Omega})$.*

(a) Suppose that $\sigma_0 < 0$. Then for any $\kappa \in (0, |\sigma_0|)$ there exists $\delta_0 > 0$ and $C > 0$ such that

$$|v_0 - u_0|_0 < \delta_0 \implies |v_{v_0}(t) - u_0|_0 \leq Ce^{-\kappa t}, \quad t \geq 0. \tag{6.1}$$

That is, u_0 is asymptotically stable.

(b) Suppose that $\sigma_0 > 0$. Then for any $\kappa \in (0, \sigma_0)$ there exists $\delta_0 > 0$ such that, for any $\delta \in (0, \delta_0)$, there exists $v_{0,\delta} \in C_0^1(\overline{\Omega})$ such that

$$|v_{0,\delta} - u_0|_1 < \delta|\psi_0|_1 \quad \text{and} \quad |v_{v_{0,\delta}}(t_\delta) - u_0|_0 \geq \delta_0,$$

where $e^{\kappa t_\delta} = 4\delta_0/\delta$. That is, u_0 is unstable.

Proof. In the case $N = 1$, Theorem 6.1 was obtained in [17, Theorem 5.1]. Having developed the above machinery of radial solutions, and the corresponding idea of a linearization, for the current case $N \geq 2$, much of the proof is now similar to the proof of [17, Theorem 5.1], so we will only describe the differences here. There are various cases to consider. We will discuss the proof of part **??** of the theorem, that is, the case $\sigma_0 < 0$, in the case $1 < p < 2$. The proof of part **??**, when $1 < p < 2$, and the corresponding proofs when $p > 2$, can be adapted from the proofs in [17] in a similar manner.

The proof is by constructing suitable sub and super-solutions of (5.1). In order to do this we first construct radial solutions η of the problem

$$\Delta_p(u_0 + s\rho + \eta) + f(u_0 + s\rho + \eta) = s\mathcal{L}_{u_0}\rho, \quad s \in \mathbb{R}, \tag{6.2}$$

for a fixed, radial function $\rho \in \mathcal{D}(\Lambda_{p,u_0})$ (recalling that u_0 satisfies (4.1)). Since we are supposing that all the functions in (6.2) are radial we may use the 1-dimensional formulation of (6.2), and we use the solution operator S_p to rewrite this in the form

$$F(s, \eta) = 0, \quad (s, \eta) \in \mathbb{R} \times C^0[0, 1], \tag{6.3}$$

where $F : \mathbb{R} \times C^0[0,1] \to C^0[0,1]$ is defined by

$$F(s,\eta) := u_0 + s\rho + \eta + S_p\big[f(u_0 + s\rho + \eta) - s\mathcal{L}_{u_0}\rho\big], \quad (s,\eta) \in \mathbb{R} \times C^0[0,1].$$

The following result now constructs solutions of (6.3).

Lemma 6.2. *For any $\rho \in \mathcal{D}(\Lambda_{p,u_0})$ there exists $\delta_\rho > 0$ and a C^1 function $\eta_\rho : (-\delta_\rho, \delta_\rho) \to C^1[0,1]$, such that:*

(a) $\eta_\rho(0) = 0$, $\partial_s \eta_\rho(0) = 0$ *(where $\partial_s \eta_0(0)$ denotes the derivative, with respect to s, of the mapping $s \to \eta_0(s) : (-\delta_0, \delta_0) \to C^1[0,1]$);*

(b) *if $|s| < \delta_\rho$ then $\eta_\rho(s)$ satisfies* (6.3)*, and so $u_0 + s\rho + \eta_\rho(s)$ is a radial solution of* (6.2)*.*

Proof. Equation (6.3) is of the same form as [17, (5.3)] (which dealt with a 1-dimensional problem), with the operator Δ_p^{-1} in [17] replaced by S_p here. The operator S_p has similar properties to those of Δ_p^{-1}, so the proof of Lemma 6.2 follows the proof of [17, Lemma 5.2]. We note that, since $\sigma_0 < 0$, it follows from Corollary 4.6 that 0 is not an eigenvalue of $\mathcal{L}_{u_0}$, which ensures that the use of the implicit function theorem in the proof in [17] is valid. $\square$

Combining part (a) of Corollary 4.6 with part (a) of Lemma 6.2 yields the following result.

Corollary 6.3. *For arbitrarily small $\beta > 0$ there exists $0 < \delta_\beta < \delta_\rho$ such that if $|s| < \delta_\beta$ then*

$$\beta s\psi_0 > |\eta_\rho(s)| \quad \textit{and} \quad \beta\psi_0 > |\partial_s \eta'_\rho(s)| \quad \textit{on } [0,1]. \tag{6.4}$$

We now construct the desired sub and super-solutions of (5.1). Since $\sigma_0 < 0$ we know that 0 is not an eigenvalue of $\mathcal{L}_{u_0}$, so the function $\zeta \in \mathcal{D}(\Lambda_{p,u_0})$ in Lemma 4.7 exists. We now write $\tau := (\delta, \gamma_1, \gamma_2) \in Q := (0,1)^3$, $|\tau| := \max\{\delta, \gamma_1, \gamma_2\}$, and we define

$$\rho_\tau := \psi_0 - \kappa\gamma_1\zeta, \quad \tau \in Q.$$

A slight extension of the proof of Lemma 6.2 (we add the variable γ_1 to the function F used in the implicit function theorem argument) shows that there exists $\delta_\rho > 0$ and, for each $\tau \in Q$ with $|\tau| < \delta_\rho$, a function η_{ρ_τ} with the properties described in Lemma 6.2 and Corollary 6.3. Hence, we may define

$$S_\tau^\pm(t) := u_0 \pm \delta e^{-\kappa t}\rho_\tau + \eta_{\rho_\tau}(\pm\delta e^{-\kappa t}) \pm \kappa\gamma_2\delta e^{-\kappa t}, \quad t \ge 0, \quad |\tau| < \delta_\rho.$$

That is, in the definition of $S_\tau^\pm$ we substitute $s = \pm\delta e^{-\kappa t}$, $t \ge 0$, into the solutions of (6.2) constructed in Lemma 6.2, and add the term $\pm\kappa\gamma_2\delta e^{-\kappa t}$. Since this latter term is constant with respect to x, it goes to zero if we apply the grad operator ∇ to it, so by part (b) of Lemma 6.2 we can apply the operator $\tilde{\Delta}_p$ to $S_\tau^\pm(t)$, and it is also clear that $S_\tau^\pm \in \tilde{\Sigma}_\infty$.

Lemma 6.4. *There exists $\tau \in Q$ such that:*

(a) $\pm(S_\tau^\pm - u_0) \ge 0$ *on Γ_∞, and there exists $\delta_0 > 0$ such that $\pm\big(S_\tau^\pm(0) - u_0\big) > \delta_0$ on $\overline{\Omega}$;*

(b) *S_τ^+ is a super-solution and S_τ^- is a sub-solution of* (5.1) *on Ω_∞.*

The proof of the above lema is essentially the same as the proof of [17, Lemma 5.4], which discusses the case $N = 1$.

Now, if τ and δ_0 are as in Lemma 6.4 then there exists $C > 0$ such that

$$|v_0 - u_0|_0 < \delta_0 \implies u_0 - Ce^{-\kappa t} \le S_\tau^-(t) \le v_{v_0}(t) \le S_\tau^+(t) \le u_0 + Ce^{-\kappa t}, \quad t \ge 0$$

(by Theorem 5.4), that is, (6.1) holds, which proves part **??** of Theorem 6.1, when $1 < p < 2$.

It remains to prove part **??** (the case $\sigma_0 > 0$), when $1 < p < 2$, and both parts part **??** and part **??** when $p > 2$ These proofs follow the corresponding proofs in [17], with only minor amendments. This finally completes the proof of Theorem 6.1. □

Remark 6.5. All the remarks in [17, Section 5.1] (entitled 'some further remarks') also hold here.

7. Bifurcating equilibria and exchange of stability

We now consider the problem

$$\frac{dv}{dt} = \Delta_p(v) + \lambda g(v)\phi_p(v), \quad v(0) = v_0 \in C_0^0(\overline{\Omega}), \tag{7.1}$$

where $\lambda \in \mathbb{R}$, $p > 2$, and $g : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a continuous, radial function, such that g_ξ exists and is continuous on $\overline{\Omega} \times \mathbb{R}$. We define

$$g_0(r) := g(r, 0), \quad g_{\xi,0}(r) := g_\xi(r, 0), \quad r \in [0, 1], \tag{7.2}$$

and we suppose that $g_0(r) > 0$, $r \in [0, 1]$.

Problem (7.1) is of the form (5.1), with $f(v) = \lambda g(v)\phi_p(v)$, and we assume that this function satisfies the boundedness condition (1.2) (given (7.2) and the form of ϕ_p, we need $p > 2$ for this to be true), so the previous results apply to (7.1). Specifically, by Theorem 5.2, (7.1) has a solution, which we will denote by $v_{\lambda,v_0} \in \Sigma_\infty$.

The equilibria of (7.1) satisfy

$$\Delta_p(u) + \lambda g(u)\phi_p(u) = 0, \quad \lambda \in \mathbb{R}, \quad u \in \mathcal{D}(\Delta_p), \tag{7.3}$$

and it is clear that (7.3) has a line of *trivial* solutions $(\lambda, 0)$, $\lambda \in \mathbb{R}$, in, say, $\mathbb{R} \times C_0^0(\overline{\Omega})$. Regarding (7.3) as a bifurcation problem, we are interested in the existence of non-trivial solutions of (7.3) bifurcating from this line of trivial solutions, and in the stability, or instability, of both the trivial solutions, and the non-trivial bifurcating solutions, when regarded as equilibria of (7.1).

If $p = 2$ and Ω is a general, smooth, bounded domain in $\mathbb{R}^N$, with $N \ge 1$, then the well-known 'simple bifurcation' results of [8] show that a curve of non-trivial solutions of (7.3) bifurcates from the line of trivial solutions at the principal eigenvalue of the linear Laplacian, and the results of [9, 12] show that there is an 'exchange of stability' between the line of trivial solutions and this curve of non-trivial solutions – see the discussion and bifurcation diagram on [12, p. 114].

If $p > 2$ and Ω is a bounded interval in $\mathbb{R}$, then it is shown in [17] that similar bifurcation and exchange of stability results hold, at the principal eigenvalue of the p-Laplacian. These results extend to the case considered here, that is, where $p > 2$ and Ω is the ball $B_1 \subset \mathbb{R}^N$, $N \ge 2$. For completeness we will summarise the results in the following theorems, but refer to [17] for further discussion and proofs. The proofs here are similar to those in [17], apart from some minor notational changes, and the occurrence of the term r^{N-1} at various points.

7.1. **Bifurcation and stability.** We begin by describing a 'simple bifurcation' result for (7.3). By [7] or [14, Lemma 1.1], the nonlinear, p-Laplacian eigenvalue problem

$$-\Delta_p(\Psi) = \lambda g_0 \phi_p(\Psi), \quad \Psi \in W_0^{1,p}(\Omega),$$

has a unique principal eigenvalue λ_0 (with $\lambda_0 > 0$) and positive, normalised, principal eigenfunction Ψ_0. Since the radial form of the eigenvalue problem has a principal eigenvalue (see [3]), it follows from the uniqueness of the principal eigenvalue λ_0 that Ψ_0 is radial. Also, by [3], $\Psi_0'(1) < 0$, so Ψ_0 is strongly positive on $[0,1)$.

Theorem 7.1. *There exists $\delta_0 > 0$ and C^1 functions $\lambda : (-\delta_0, \delta_0) \to \mathbb{R}$, $y : (-\delta_0, \delta_0) \to C^0[0,1]$, such that, for each $s \in (-\delta_0, \delta_0)$, the following results hold.*

(a) $(\lambda(s), u(s)) = (\lambda(s), s\tilde{u}(s)) := (\lambda(s), s(\Psi_0 + y(s)))$, is a radial solution of (7.3).

(b) $\lambda(0) = \lambda_0$, $\quad y(0) = 0$, $\quad \langle y(s), \Psi_0 \rangle = 0$, and

$$\lambda'(0) = -\lambda_0 \frac{\langle r^{N-1} g_{\xi,0}, \Psi_0^{p+1} \rangle}{\langle r^{N-1} g_0, \Psi_0^{p} \rangle}. \tag{7.4}$$

(c) The function $y : (-\delta_0, \delta_0) \to C_0^1(\overline{\Omega})$ is continuous, so $\tilde{u}(s)$ is strongly positive or strongly negative, with $\operatorname{sgn} u(s) = \operatorname{sgn} s = \lambda'(0) \operatorname{sgn}(\lambda(s) - \lambda_0)$.

(d) There exists a neighbourhood U_0 of $(\lambda_0, 0)$ in $\mathbb{R} \times C_0^0(\overline{\Omega})$ such that the set of all radial, non-trivial solutions of (7.3) in U_0 consists of the set $\{(\lambda(s), u(s)) : 0 < |s| < \delta_0\}$.

(e) The principal eigenvalue of the linearization at $u(s)$ has the form $|s|^{p-2}\tilde{\sigma}_0(s)$, where the function $\tilde{\sigma}_0 : (-\delta_0, \delta_0) \to \mathbb{R}$ is continuous, and is differentiable at $s = 0$, with $\tilde{\sigma}_0(0) = 0$ and

$$\tilde{\sigma}_0'(0) = \lambda_0 \frac{\langle r^{N-1} g_{\xi,0}, \Psi_0^{p+1} \rangle}{\langle r^{N-1} \Psi_0, \Psi_0 \rangle}. \tag{7.5}$$

(f) If the numerator in the expressions in (7.4) and (7.5) is non-zero then

$$\operatorname{sgn} \tilde{\sigma}_0'(0) = -\operatorname{sgn} \lambda'(0) \neq 0.$$

Proof. Most of the results in Theorem 7.1 were obtained in [15, Theorem 4.1 and Corollary 4.5], although similar results had been obtained in several previous papers; see the proof of [17, Theorem 6.1] for further discussion of this. In the case $N = 1$, the formulae (7.4) and (7.5) were obtained in Theorems 6.2 and 6.5 (respectively) in [17]; the proofs of these results in the present setting are identical. $\square$

The bifurcating solutions found in Theorem 7.1 are radial (they are explicitly constructed from the radial form of the problem), and the theorem finds all radial solutions in the neighbourhood U_0 referred to in part (d). However, this does not rule out the possibility of the existence of non-radial solutions in U_0. The conditions described in Remark 4.1 above would rule out such non-radial solutions.

Theorem 7.1 obtained non-trivial, bifurcating solutions, and parts (e) and (f) determined their linearized (and hence, dynamic) stability. The following theorem describes the stability of the trivial solutions.

Theorem 7.2. *(a) If $\lambda < \lambda_0$ then the trivial solution $(\lambda, 0)$ of (7.1) is asymptotically stable.*

(b) If $\lambda > \lambda_0$ then the trivial solution $(\lambda, 0)$ of (7.1) is unstable.

7.2. **Exchange of stability.** Combining Theorem 7.1, in particular, part (f), and Theorem 7.2 now yields exchange of stability for the trivial and bifurcating, non-trivial solutions (in a neighbourhood of $(\lambda_0, 0)$ in $\mathbb{R} \times C_0^0(\overline{\Omega})$), which can be summarized as follows.

- The trivial solutions $(\lambda, 0)$ are stable when $\lambda < \lambda_0$ and unstable when $\lambda > \lambda_0$.
- If the numerator in the expressions in (7.4) and (7.5) is non-zero then the bifurcating solutions (λ, u) are stable when $\lambda > \lambda_0$ and unstable when $\lambda < \lambda_0$. The sign of u in these solutions is determined by part (c) of Theorem 7.1 (positivity of u may be relevant in applications).

Thus, we obtain a similar bifurcation diagram to that on [12, p. 114], which considered the case $p = 2$. The idea of exchange of stability of bifurcating solutions is well-known, at least when $p = 2$, so we will not discuss this further.

7.3. **Curves of equilibria, saddle-node bifurcations and stability.** A common technique to obtain (equilibrium) solutions of (7.3), and determine their exact multiplicity at specific values of λ, is to find curves of solutions in, say $\mathbb{R} \times C_0^0(\overline{\Omega})$ (that is, find the bifurcation diagram), and then determine the 'shape' of this curve (that is, determine the number of turning points of the curve in the bifurcation diagram). An archetypal example of this procedure is discussed in [13] for a semilinear ($p = 2$) problem, where it is shown that the solution curve obtained there has exactly two turning points (the bibliography of [13] contains many other papers which adopt this procedure). We refer to [13] for more details of this procedure and the terminology used – here we will simply give a brief, heuristic description of the procedure and its connection with the above linearized stability results.

The turning points on the solution curves are usually 'saddle-node' or 'fold' bifurcation points, and they correspond to points at which the principal eigenvalue of the linearization operator at the solutions on the curve changes sign as we move along the curve (see [13]). Hence, the above procedure determines the sign of the principal eigenvalue at all points along the curve and so, if the principle of linearized stability holds, it determines the stability or instability of the solutions on the curve. This is well known in the semilinear case (where the principle of linearized stability holds), but we can now hope to extend the stability results to the quasilinear, p-Laplacian problem (7.3), given that we have established a principle of linearized stability for this problem.

Unfortunately, it is also difficult to obtain the shape of the solution curves in the quasilinear case, due to the difficulty in obtaining the required differentiability. In fact, second order differentiability is usually required to describe the turning points of the curve, and even first order differentiability is difficult to obtain in the quasilinear case. However, a p-Laplacian problem of the type considered here was discussed in [1] (with $1 < p < 2$ and $N > p$), where a solution curve with exactly one turning point was obtained. Combining Theorem 6.1 above with the constructions in [1] (which obtained the sign of the principal eigenvalue of the solutions on the curve) shows that the solutions on the 'lower' branch of the solution curve in the bifurcation diagram found in [1] are stable, while the solutions on the upper branch are unstable. Similar results on the shape of the solution curve, and the stability of the solutions on this curve, were obtained in [16] for the case $N = 1$.

Given similar results on curves of equilibrium solutions of (7.3) for other problems, the above linearized stability results would determine the stability or instability of these solutions. However, determining such curves of solutions is another matter, so we will not say any more here.

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